

Decoherent histories on graphs

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February 7, 2008

Abstract

The consistent histories approach to quantum mechanics is traditionally based on linearly ordered sequences of events. We extend the histories formalism to sets of events whose causal ordering is described by directed acyclic graphs. The need for a global time is eliminated and our construction reflects the causal structure faithfully.

1 Introduction

The *consistent histories* approach to quantum mechanics due to Griffiths and Omnès [4, 6] was formulated with the aim of shedding new light on the conceptual difficulties of the theory. A closely related proposal with different motivation is the *decoherent histories* approach to quantum cosmology of Gell-Mann and Hartle [3]. The basic ingredient in both approaches is the notion of a *history* of the quantum system described by a sequence of projection operators in the Hilbert space of the system for a succession of times. The goal of quantum mechanics is to determine the probability of an event or a sequence of events, thus one might hope to assign probabilities to the histories of the quantum system. The probabilities have to be additive for histories describing mutually exclusive possibilities. Sets of histories obeying this consistency condition are selected with the use of a special bilinear form on histories - the decoherence functional. Families of histories consistent with respect to the decoherence functional are then unambiguously assigned probabilities. An excellent exposition of these ideas is contained in [4].

It is customary to represent the individual histories mathematically as linearly ordered sequences of projection operators in the Hilbert space of the quantum mechanical system.

*Research supported in part by NSERC.

But the linear causal ordering of the events in a history is too restrictive in many experimental situations, in particular when analyzing spatially separated entangled quantum systems. This issue is even more pressing for quantum cosmology considerations. An application of the histories approach to quantum field theory on a curved space-time [1] also assumes the existence of a globally hyperbolic manifold with the ensuing linear ordering of events in a history. The basic ideas of our proposal for describing the evolution of an open quantum system [2], could also be used to describe a single history in a set of histories of a closed quantum system. In our scheme the events are no longer required to be linearly ordered with respect to the causal order. There is no global time and the causal relations between events are described by graphs generalizing the causal sets of [7]. Most importantly, the consistency/decoherence condition for histories has an immediate generalization for histories described by more general graphs as proposed here.

2 Quantum evolution on graphs

2.1 Kinematics

A description of the history of a quantum system consisting of several spatially separated subsystems must include the description of the causal relations between different events in space-time. These causal relations will be represented by graphs with the events at the vertices and the edges representing causal influences. These influences are propagated by parts of the system - with their own quantum degrees of freedom - traveling from one space-time point to another. Thus every edge will be labelled with a Hilbert space accounting for these local degrees of freedom. Moreover, every edge will also carry a density matrix in the Hilbert space of the edge, describing the knowledge that a local observer has about the quantum subsystem associated with the edge. With every vertex v_i of the graph two Hilbert spaces are associated naturally. The tensor product of all Hilbert spaces on the incoming edges \mathcal{H}_i^{in} and similarly the outgoing Hilbert space of the vertex \mathcal{H}_i^{out} . For every vertex, the incoming and outgoing Hilbert spaces will have the same dimension¹ and will be identified $\mathcal{H}_i^{in} \cong \mathcal{H}_i^{out} = \mathcal{H}_i$.

The vertices will represent events of two types. First a subsystem of our quantum system could undergo a local unitary evolution. Vertices of the graph representing such events will be labelled with unitary operators. A unitary operator U_i at a vertex v_i acts on \mathcal{H}_i . The second type of vertices will be labelled with projection operators in the corresponding Hilbert space. These vertices depict the fact that a particular property of a subsystem is realized at the corresponding point in space-time in the history being described by the graph. Consider for example the simple graph of Figure 1.

The vertices are drawn as boxes with the associated unitary or projection operators inside. The graph depicts the history of a quantum system prepared as two separate subsystems on the edges e_a and e_b . The e_a subsystem undergoes a unitary evolution with an operator U_1 and then splits in two along the edges e_c and e_d . The e_b subsystem realizes the property described by the projection operator P_2 before coming together and interacting with the e_d

¹In contrast with the more general considerations of [2].

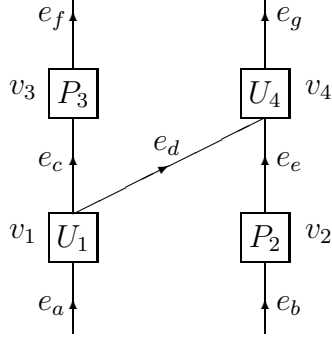


Figure 1:

subsystem at v_4 . The subsystem e_c realizes the property described by the projector P_3 . The events U_1 and P_2 (or P_3 and P_2) are causally unrelated and thus it makes no sense to say that one occurs before or after the other.

More complicated quantum systems will be described by more complicated graphs with the following two properties. The graphs will be directed, reflecting the direction of the causality relation, and they will be acyclic thus excluding any temporal loops. Notice also that although the processes of unitary evolution take a certain amount of time, we are only interested in the causal relations between events and this allows us to consider them as pointlike vertices on the graph. Thus we are thinking of the duration between events as being longer than the duration of an event so that no causal information is lost when we represent interactions as points.

The causal relations specified by a directed acyclic graph (*dag*) is described as follows. A vertex v_i is in the future of a vertex v_j iff there is a directed path of oriented edges starting at v_j and ending with v_i . In this case we also say that v_j is in the past of v_i . An edge e_i is in the future of an edge e_j if the initial vertex of e_i is in the future of the final vertex of e_j . Future and past relations between a vertex and an edge are defined similarly. An edge is initial (final) if it has empty past (future). Two edges (vertices) which are not causally related will be called *acausal*. A set of acausal edges will be called a *slice*. Note that the initial (or final) edges form a slice. We call it the initial (final) slice.

So far we have the causal kinematics of a quantum system encoded in a directed acyclic graph (*dag*). To describe the dynamics we have to describe how the operators at the vertices act on the density matrices living on the edges and how these actions compose to describe the evolution of the system. Several requirements have to be satisfied by such a prescription. Since the graph is supposed to reflect the causality relation, there should be no influences across the graph breaking causality. For example, for the graph of Figure 1 the density matrix on the edge e_f should not depend on the operators P_2 or U_4 . To respect causality the action of the operators should be local, i.e. in the Hilbert spaces associated with the vertex, but it cannot be too local, otherwise our description will not reflect the possible entanglement between subsystems associated with different edges. To define the action of the operators at the vertices we need to associate density matrices not only with single edges, but more generally with slices. Notice that the set of incoming (outgoing) edges at a

vertex are acausal and thus form a slice. The density matrix associated with a slice could be defined to be the tensor product of the density matrices of the edges which form the slice, but it is easy to see on the example of a graph representing a simple EPR situation that this prescription cannot account dynamically for the existence of entanglement between spatially separated subsystems (represented here by the acausal edges of the slice). Thus density matrices genuinely live on the slices of the graph; in general the density matrix on a slice is not equal to the tensor product of the density matrices on the edges forming the slice.

If a slice L consisting of the acausal edges e_1, \dots, e_l is embedded in a slice M consisting of the edges $e_1, \dots, e_l, e_{l+1}, \dots, e_m$ then its Hilbert space $\mathcal{H}_L = \mathcal{H}_{e_1} \otimes \dots \otimes \mathcal{H}_{e_l}$ embeds in $\mathcal{H}_M = \mathcal{H}_{e_1} \otimes \dots \otimes \mathcal{H}_{e_m}$. In accordance with the physical intuition the density matrix ρ_L associated with L is obtained from the density matrix ρ_M via a partial trace: $\rho_L = \text{Tr}^{l+1, \dots, m} \rho_M$. In particular the density matrix on an edge could be obtained from the density matrix of any slice containing the given edge by a partial trace in the Hilbert space of the slice.

2.2 The dynamical prescription

We are now ready to start discussing the dynamics of a quantum system represented by a dag G . Dynamics will be described by supposing that we are given a density matrix on the initial spacelike slice, and then giving a prescription for calculating the density matrices of future spacelike slices. In essence, we are propagating the initial data throughout the system.

To each vertex $v_i \in G$ will be assigned an operator T_i . Let ρ_i^{in} be the density matrix associated to the slice of incoming edges at v_i . Then one obtains the density matrix for the slice of outgoing edges by:

$$\rho_i^{out} = T_i(\rho_i^{in}).$$

Here we used the fact that the set of incoming (outgoing) edges to a vertex is acausal and thus forms a slice. Notice that more generally, for two acausal vertices, the sets of incoming or outgoing edges are pairwise acausal. Thus, the associated operators will act on distinct Hilbert spaces and hence commute when extended to act in the product Hilbert space. Without loss of generality, we also follow the convention that if the initial slice consists of several edges, the initial state of the whole system is a tensor product state, i.e. the subsystems corresponding to the initial edges are not entangled. Entangled subsystems on distinct edges will always have at least one event in the common past.

We begin with an illustrative example. Consider the dag of Figure 2. Given the state on the initial slice, the operators at the events propagate the state to the future. In the example of Figure 2 we have: $\rho_c = T_1(\rho_a)$, $\rho_{fde} = T_2(\rho_b)$ ². However the next operator T_3 must act on the so far undefined density matrix ρ_{cd} . T_3 takes density matrices on $\mathcal{H}_c \otimes \mathcal{H}_d$ to those on $\mathcal{H}_g \otimes \mathcal{H}_h$. By extending T_3 with the appropriate identity operators, we can view it as a map from $\text{DM}(\mathcal{H}_c \otimes \mathcal{H}_d \otimes \mathcal{H}_e \otimes \mathcal{H}_f)$ to $\text{DM}(\mathcal{H}_e \otimes \mathcal{H}_f \otimes \mathcal{H}_g \otimes \mathcal{H}_h)$. Here $\text{DM}(-)$ denotes the space of density matrices on a Hilbert space. Then we can define the density matrix on another spacelike slice, namely $\rho_{fgh} = T_3(\rho_c \otimes \rho_{fde})$. Similarly $\rho_{fdi} = T_4(\rho_{fde})$ and so on.

²We decorate the density matrix on a slice with the labels of the edges which constitute the slice.

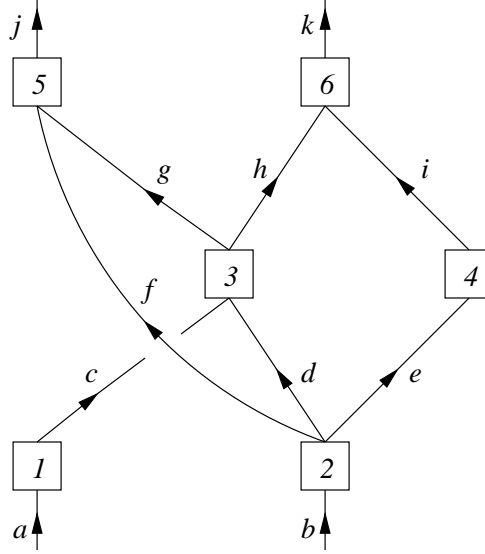


Figure 2:

Starting from density matrices on the initial edges and using the operators associated with the vertices - extended with identities as needed - we obtain density matrices on specific spacelike slices.

The above inductive process for propagating density matrices can be applied to any system described by a dag. However, the procedure only gives the density matrices for certain spacelike slices within the dag. For example, this procedure does not yet yield a matrix for the slice de . To calculate such density matrices, we will also have to make use of the trace operator. Before extending the procedure to general slices, we first consider those for which the above process is sufficient. We call these slices *locative*.

Definition 2.1 Let G be a dag, and L a slice of G . Consider the set of all vertices V which are to the past of some edge in L . Let I be the set of initial edges in the past of L . Consider all paths of maximal length beginning at an element of I and only going through vertices of V . Then L is *locative* if all such paths end with an edge in L .

In our example, the locative slices are the following:

$$a, b, ab, c, cb, def, adef, cdef, efgh, adfi, cdfi, fghe, fghi, fgk, hej, hij, jk$$

while, for example, de is not locative. Note that the fact that maximal slices are always locative follows immediately from the definition of locative.

We now describe the general rule for calculating the density matrices on locative slices. Associated with each locative slice L is the set I of initial edges in the past of L . We choose a family of slices that begins with I and ends with L in the following way. Consider the set of vertices V between the edges in I and the edges in L . Because L is locative we know that propagating slices forwards through the vertices in V will reproduce L . Let $M \subset V$ be such that the vertices in M are minimal in V with respect to causal ordering. We choose

arbitrarily any vertex u in M , remove the incoming edges of u and add the outgoing edges of u to the set I obtaining a new set of edges I_1 . It is clear that I_1 is spacelike and locative. Proceeding inductively in this fashion we obtain a sequence of slices $I = I_0, I_1, I_2, \dots, I_n = L$, where n is the cardinality of V . Of course, this family of slices is far from unique.

The dynamics is obtained as follows. Recall that the states on initial edges are assumed not to be entangled so that one can obtain the density matrix on any set of initial edges, in particular I , as a tensor product. Let ρ_0 be the density matrix on I . We look at the vertex u that was used to go from I to I_1 and apply the operator T assigned to this vertex - possibly augmented with identity operators as in the example above. Proceeding inductively along the family of slices, we obtain the density matrix ρ_n on L .

The important point now is that ρ_n does not depend on the choice of slicing used in going from I to L . This can be argued as follows. Suppose we have a locative slice S and two vertices u and v which are both causally minimal above S and acausal with respect to each other. Then we have four slices to consider, S , S_u , S_v and S_{uv} where by S_u we mean the slice obtained from S by removing the incoming edges of u and adding the outgoing edges of u to S and similarly for the others. It is clear, in this case, that the operators assigned to u and to v commute and the density matrix computed on S_{uv} is independent of whether we evolved along the sequence $S \rightarrow S_u \rightarrow S_{uv}$ or $S \rightarrow S_v \rightarrow S_{uv}$. Now when we constructed our slices at each stage we had the choice between different minimal vertices to add to the slice. But such vertices are clearly pairwise acausal and hence, by the previous argument applied inductively, the evolution prescription is independent of all possible choices.

So far we have defined density matrices on locative slices only. To define density matrices on general spacelike slices, we will need to consider partial tracing operations.

2.3 General Slices

Recall if a quantum system Q consists of two subsystems Q_1 and Q_2 , the Hilbert space for Q may be decomposed as $\mathcal{H}_1 \otimes \mathcal{H}_2$ where \mathcal{H}_i represents Q_i . The density matrix for Q_1 is obtained from the density matrix for Q by tracing over \mathcal{H}_2 . To obtain a candidate for the density matrix of a slice L , we should find a locative slice M that contains L and trace over the Hilbert spaces on edges in $M \setminus L$. Such a locative slice M always exists because maximal slices are always locative. M is not unique however, and thus - as we did for locative slices - we must show that different choices give the same result. To simplify the notation we will discuss the case of density matrices associated with single edges. The case of a general spacelike slice is similar.

Consider an edge e_i in a graph G . Let $V_i = \{v_{i_1}, \dots, v_{i_p}\}$ be the set of vertices in the past of e_i . Let $I_i = \{e_{i_1}, \dots, e_{i_q}\}$ be the set of initial edges in the past of e_i . Constructing a sequence of slices by incrementally incorporating the vertices of V_i in a manner similar to what we did in the previous subsection, we get a locative slice M_i containing e_i . Starting with the density matrices on the edges of I_i and applying the operators associated with the vertices of V_i , we obtain the density matrix on the locative slice M_i . It is clear that M_i is in an evident sense the minimal locative slice containing e_i .

Definition 2.2 We shall refer to M_i as the *least locative slice* of the edge e_i .

Let the least locative slice M_i of an edge e_i consist of edges $\{e_i, e_{j_1}, \dots, e_{j_r}\}$. The density matrix ρ_{i,j_1,\dots,j_r} on M_i is an element of the space $End(\mathcal{H}_i \otimes \mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_r})$. Let $Tr^{j_1 \dots j_r}$ be the partial trace operation $End(\mathcal{H}_i \otimes \mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_r}) \rightarrow End(\mathcal{H}_i)$.

Definition 2.3 (Density matrix associated with an edge) The density matrix ρ_i at the edge e_i is defined to be:

$$\rho_i = Tr^{j_1 \dots j_r} \rho_{i,j_1,\dots,j_r}. \quad (1)$$

If M_i consists of the single edge e_i , then no tracing is done.

Remark 2.4 *The causality condition for evolving the initial data on G requires that the density matrix associated with a given edge e_i depends only on the initial data in the past of e_i and only those interventions to the past of e_i . The density matrix ρ_i as defined in 2.3 satisfies this requirement by construction and so our prescription for dynamical evolution is causal.*

In general, the edge e_i is contained in many locative slices and we could just as well have defined ρ_i by tracing over the complimentary degrees of freedom in any of these locative slices. Independence of the resulting density matrices is the discrete analog of Lorenz (or general) covariance in our framework. A detailed discussion of covariance will take us too far afield and we refer the interested reader to [2].

3 The consistency condition

The scheme for describing quantum evolution of spatially separated entangled quantum systems by discrete graphs presented in the previous section allows for considerable strengthening of the language of consistent histories. In particular it permits reasoning about spatially separated quantum systems which may or may not be entangled. Moreover there is no total temporal ordering of the events, only the causal relations which are independent of any external observer are tracked in a history. Consider the example of Figure 3.

The history of the system starts with three unentangled spatially separated subsystems on the edges a, b and c . The subsystem labelled by a undergoes a unitary evolution and then splits into two subsystems d and e . The subsystem labelled by b realizes a certain property represented by the projection operator P_2 and then splits into three subsystems. The subsystem c combines with the subsystem h and realizes the property represented by P_3 . The history continues to unfold with events at vertices 4 and 5. Thus we can track the evolution and properties of spatially separated subsystems without the need for a global choice of time. The operators at the events act locally on the Hilbert spaces associated with the corresponding vertices. These Hilbert spaces are:

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{H}_a = \mathcal{H}_d \otimes \mathcal{H}_e \\ \mathcal{H}_2 &= \mathcal{H}_b = \mathcal{H}_f \otimes \mathcal{H}_g \otimes \mathcal{H}_h \\ \mathcal{H}_3 &= \mathcal{H}_k = \mathcal{H}_c \otimes \mathcal{H}_h \\ \mathcal{H}_4 &= \mathcal{H}_i = \mathcal{H}_d \otimes \mathcal{H}_f \\ \mathcal{H}_5 &= \mathcal{H}_j = \mathcal{H}_e \otimes \mathcal{H}_g \end{aligned}$$

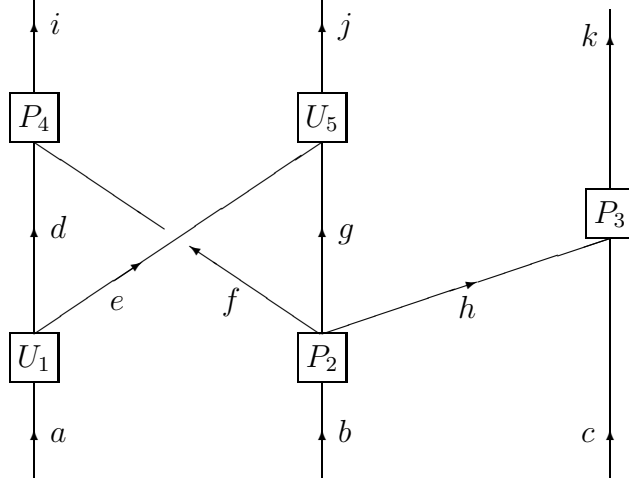


Figure 3:

The labelled graph is a representation of a history, and the evolution operator associated to the graph is a map from density matrices to density matrices; a density matrix is, of course an endomorphism on a Hilbert space. Thus, in the above example, the evolution operator T will be of type

$$T : End(\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c) \rightarrow End(\mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_k).$$

The initial Hilbert space associated with the incoming subsystems is isomorphic to the final Hilbert space associated with the outgoing subsystems; i.e. $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ and $\mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_k$ are essentially the same Hilbert space. Thus, writing \mathcal{H} for this space, we see that the evolution operator has type $End(\mathcal{H}) \rightarrow End(\mathcal{H})$.

Consider a labelled directed acyclic graph G representing a history of a quantum system. Some of the vertices of the graph are labelled with projection operators, and we call these *property vertices*. The remaining vertices of the graph are labelled with unitary operators and we call them *evolution vertices*. Suppose G has n property vertices v_1, \dots, v_n labelled with projection operators representing properties of specific subsystems. In order to easily express the action of these projection operators as a tensor product, we arbitrarily choose a linear ordering of these n vertices. However, note that this ordering should in no way be thought of as necessarily related to the causal ordering. It is simply a notational convenience. To the graph G , we can now associate an operator

$$P_G = P_1 \otimes \dots \otimes P_n \in (Proj\mathcal{H}_1) \otimes \dots \otimes (Proj\mathcal{H}_n),$$

where $Proj\mathcal{H}_i$ is the space of projections of the Hilbert space at the vertex v_i .

Composing the operators (both unitary and projectors) at the vertices of G according to the prescription of section 2.2 provides an evolution operator which takes the initial density matrix to the final density matrix. If we think of the projection operators as unspecified, then when we compose we get a functional that depends on the projection operators:

$$K : (Proj\mathcal{H}_1) \otimes \dots \otimes (Proj\mathcal{H}_n) \mapsto End\mathcal{H},$$

where \mathcal{H} is the Hilbert space of the initial (or equivalently the final) slice of G . The value of K at $P_1 \otimes \cdots \otimes P_n$ is the evolution operator. It is given by composition of all operators at the vertices of the graph. The definition of the operator K depends on the operators labelling the evolution vertices of G , but notationally we have suppressed this dependence.

Consider the specific example of Figure 3. Suppose that our initial density matrix is ρ_{abc} . Then this density matrix propagates to the final density matrix ρ_{ijk} by the formula:

$$\rho_{ijk} = U_5 P_4 P_3 P_2 U_1 (\rho_{abc})$$

But note that when two vertices are spacelike separated, the corresponding operators commute. As discussed at length in section 2.2, this leads to the possibility that there will be many equal expressions for the final density matrix. For example, we also have:

$$\rho_{ijk} = P_4 U_5 U_1 P_3 P_2 (\rho_{abc})$$

The operator K plays an analogous role to the operator K in the consistent histories approach [4]. There K acts on histories which are time-labelled sequences of projection operators, whereas here K acts on a family of projectors labelling a directed acyclic graph. In our language, a history in the sense of [4] will be represented by a linear order as in Figure 4. Thus if we restrict our posets to such linear orders we obtain the usual consistent histories approach.

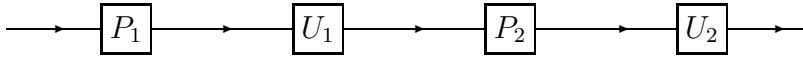


Figure 4:

Consider now a family $\{G_\alpha\}$ of labelled dags. Each labelled graph G_α in the family has the same underlying graph G and the same unitary operators at their corresponding evolution vertices. The projection operators however might differ and to every G_α we associate an operator $P_\alpha = P_{\alpha 1} \otimes \cdots \otimes P_{\alpha n}$, where $P_{\alpha i}$ is the operator labelling the vertex v_i in the graph G_α . The operators P_α generate an associative algebra under pointwise multiplication and addition of projection operators. We think of every such labelled graph G_α as representing a particular history and abusing the notation we will call the operators P_α histories as well, although they contain only partial information from G_α .

Consider a family $\{G_\alpha\}$ of labelled dags, with fixed underlying graph and unitary operators, such that the operators P_α obey

$$\sum_{\alpha=1}^n P_\alpha = I, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha,$$

that is the operators P_α form an orthogonal decomposition of the identity of the Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$. The conditions above guarantee that every possible evolution of the

quantum system with the same unitary operators, but with possibly different properties being realized along the way, is contained in precisely one history G_α . We call such a family of labelled dags a *family of histories*. Taking linear combinations of histories $Y = \sum_\alpha t_\alpha P_\alpha$, with coefficients t_α which are either 0 or 1, we can form a Boolean algebra of histories based on the same underlying graph.

For a history P_α in a family of histories we define its *weight* to be

$$W(P_\alpha) = \text{Tr}[K(P_\alpha)^\dagger K(P_\alpha)].$$

$W(P_\alpha)$ is the (unnormalized) probability for the history G_α to be realized starting from initial state described by a density matrix equal to identity on the initial Hilbert space. The *consistency condition* then requires that the probabilities add for histories that are mutually exclusive:

$$P_\alpha P_\beta = 0 \quad \text{must imply} \quad W(P_\alpha + P_\beta) = W(P_\alpha) + W(P_\beta).$$

Using the definition of a weight of a history, it is immediate that the consistency condition requires that in a family of histories, one has

$$\text{Re Tr}[K(P_\alpha)^\dagger K(P_\beta)] = 0 \quad \text{for } \alpha \neq \beta.$$

The stronger condition $\text{Tr}[K(P_\alpha)^\dagger K(P_\beta)] = 0$ for $\alpha \neq \beta$ is often also considered. Following [4] we call a family of histories (or the corresponding Boolean algebra) satisfying the (strong) consistency condition a *framework*. The underlying graph and the labels of the evolution vertices are the same for all the individual histories in the framework.

Reasoning about the quantum system and its spatially separated subsystems must start with the selection of an appropriate framework. In particular the probability for a given history will depend on the framework in which it is considered. To illustrate the quantum reasoning based on consistent histories, we now consider the notion of refinement.

4 Refinements

One application of the consistent histories approach is to analyze experiments performed on quantum systems. Some of the information about the quantum system under investigation comes with the experimental apparatus and is predefined (known with certainty). For example, the complete state of the quantum system might be known at a given starting time. We can consider more general experimental setups, where the setup data is described by exclusive possibilities, each one of these possibilities being assigned a predefined probability. For example, it might be known that an electron beam has probability 1/2 of having its spins pointing in the positive direction of the z axis. To describe the data already known, we can utilize a framework of histories, each individual history describing one possible experimental setup. To describe the questions asked about the quantum system and the possible outcomes, we need the notion of refinement.

We consider two general cases of refinement. First, each framework comes with an underlying graph and we can refine the family at an already existing vertex. For example a

particular history in the family, $P_\alpha = P_1 \otimes \cdots \otimes P_i \otimes \cdots \otimes P_n$ could be split into two mutually exclusive histories $P_{\alpha 1} = P_1 \otimes \cdots \otimes P_{i1} \otimes \cdots \otimes P_n$ and $P_{\alpha 2} = P_1 \otimes \cdots \otimes P_{i2} \otimes \cdots \otimes P_n$, where $P_i = P_{i1} + P_{i2}$, $P_{i1}P_{i2} = 0$. The new framework will contain the two new histories $P_{\alpha 1}$ and $P_{\alpha 2}$ in place of P_α . This form of refinement allows us to ask more detailed questions about the quantum system at a space-time point which is already targeted for examination.

Another possibility for refining a framework is to blow up an edge of the underlying graph. Each edge e_i of the graph represents a quantum subsystem traveling along this edge

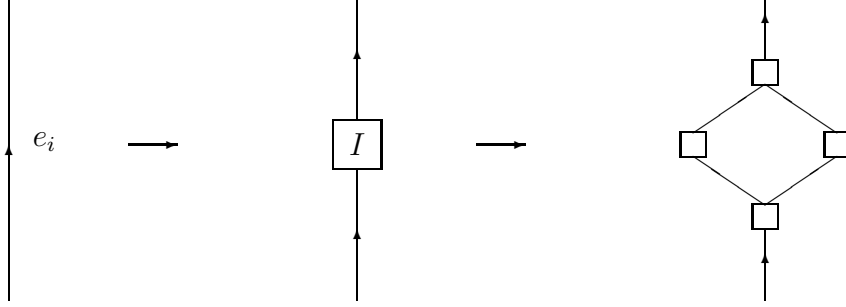


Figure 5:

undisturbed. A graph which has a box with identity operator inserted at this edge describes the same history. Now we can decompose this identity operator into smaller projectors to refine the history and also the framework. More generally, the identity operator at the edge e_i can be refined by a graph with one initial and one final edge. Figure 5 provides a pictorial example. The refined framework of histories will have a different underlying graph, one or more edges of the underlying graph of the unrefined framework being replaced with new subgraphs with one initial and one final edge. The histories of the original framework could be extended by identity operators at the new vertices. These identity operators at the new vertices then can be split into smaller projectors according to the first scheme for refinement. Blowing up edges allows us to single out new space-time points for questioning the quantum system.

The reasoning about the quantum system in the histories framework then proceeds as follows [4]. The initial framework \mathcal{F} and probabilities for the individual histories are assigned on the basis of the data known about the quantum system. Formally, we suppose given an initial framework $\mathcal{F} = \{P_\alpha\}$, where P_α is an individual history. Then a probability distribution on it will be an assignment of probabilities $Pr(-)$ to the individual histories in such a way that $Pr(P_\alpha) \geq 0$, $\sum_\alpha Pr(P_\alpha) = 1$, $W(P_\alpha) = 0 \Rightarrow Pr(P_\alpha) = 0$. The last condition requires that histories which are dynamically impossible are assigned zero probabilities. We start with a framework \mathcal{F} and a probability distribution on it. The further questions about the quantum system will be expressed in a framework \mathcal{G} which is a refinement of \mathcal{F} . Given an individual history Y in \mathcal{G} we assign its probability by

$$Pr(Y) = \sum_\alpha W(Y|P_\alpha) Pr(P_\alpha).$$

Here $W(Y|P_\alpha) = W(Y P_\alpha)/W(P_\alpha)$ is the conditional probability for Y occurring given P_α . To form the product $Y P_\alpha$ the history P_α might need to be extended by identity operators at

certain points of the new graph underlying \mathcal{F} , as explained above. Notice that the probability for the histories in \mathcal{G} which belonged to the initial framework \mathcal{F} remains unchanged.

This mode of quantum reasoning is as follows. The initial probability distribution on some framework, is initial not in time, but rather encodes what is known about the quantum system already. It serves as a starting point for further questions and valid conclusions about the quantum system. The questions which could be asked are of the type: has a subsystem realized a given property at a given space-time point and an answer will be given by a probability computed for the corresponding refined history. Our scheme allows for questions about spatially separated quantum subsystems such as questions localized in space as well as time. Dynamical evolution itself could be described by refinement, namely refinement on one or more of the final edges of the graph. The answers, i.e. the probabilities for individual histories, will in general depend on the refined framework they are members of, since by the Kochen-Specker theorem it is impossible to consistently assign truth values to the projection operators in a Hilbert space of dimension bigger than two. Thus it is impossible in general to put two frameworks together even if they have the same underlying graphs and evolution operators. Questions in quantum mechanics always come with their context (framework) and the answers we get depend on the context.

5 Conclusions

We have presented a scheme for describing closed quantum systems which extends the consistent/decoherent histories approach to quantum mechanics. The evolution is local and causality is made explicit in the description. Crucially, the individual histories in our framework are detailed enough to describe properties of spatially separated subsystems, without sacrificing the ubiquitous entanglement. In particular, the linear ordering of the events in a history is no longer necessary, nor is a global notion of time. Our approach allows for the consideration of questions localized in time as well as space.

We note that the composition of the operators representing the events can be encoded in a mathematical structure called a *polycategory*. The algebraic and logical aspects of our scheme are discussed in more detail in [2].

Acknowledgements

The authors would like to thank NSERC for its financial support. We would also thank Rafael Sorkin for a lengthy discussion on causal sets. The paper [5], which led to our initial consideration of these ideas, was pointed out to us by Ioannis Raptis. Finally, the second author would like to thank the University of Ottawa Department of Mathematics for its hospitality and support.

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